Exercise 26

Prove Equation 4.

Solution

The aim is to prove Equation 4 on page 262.

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right).$$

Let

$$y = \cosh^{-1} x.$$

Then

$$\cosh y = x. \tag{1}$$

Use the result from Example 1(a), $\cosh^2 y - \sinh^2 y = 1$, to get a formula for $\sinh y$.

$$\sinh y = \pm\sqrt{\cosh^2 y - 1}$$

The hyperbolic cosine is not a one-to-one function because it fails the horizontal line test, so it's necessary to take the restriction to nonnegative arguments in equation (1) for an inverse function to exist. In other words, y is not negative, so $\sinh y$ also is not negative.

$$\begin{aligned}\sinh y &= \sqrt{\cosh^2 y - 1}\\ &= \sqrt{x^2 - 1}\end{aligned}$$

According to Exercise 9, $\cosh y + \sinh y = e^y$, so

$$\begin{aligned}e^y &= \cosh y + \sinh y\\ &= x + \sqrt{x^2 - 1}.\end{aligned}$$

Take the natural logarithm of both sides to solve for y.

$$\begin{aligned}\ln e^y &= \ln\left(x + \sqrt{x^2 - 1}\right)\\ y \ln e &= \ln\left(x + \sqrt{x^2 - 1}\right)\\ y &= \ln\left(x + \sqrt{x^2 - 1}\right)\end{aligned}$$

Therefore,

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right).$$